

**NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS**

TECHNICAL NOTE 2494

**LIFT AND MOMENT ON OSCILLATING TRIANGULAR AND
RELATED WINGS WITH SUPERSONIC EDGES**

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Page 4, line 3: Replace the quantity $M^2 - 1$ by $\sqrt{M^2 - 1}$.

Page 11, equation (17): In the first value for μ_1 for plan form D,
the term $\frac{\tau}{2b}$ within the final parentheses should be $\frac{l}{2b}$.

Page 21: Correct equations (B3) as follows:

In line 2, the expression $\alpha_5 - E_{21}$ should be $\alpha_5 E_{21}$.

In lines 4, 6, 7, 13, 15, and 18, all primes appearing on the right-hand side of these equations should be replaced by the superscript 1.

In line 14, the sixth term of the equation for $-2\pi M_3'$ should be $\delta_3 H_{12}^0$ instead of $\delta_3 H_{12}$.

Page 22: In line 5, the " M_1^2 " within the bracketed expression should be " M^2 ."

In line 7: The right-hand side of the equation for β_3 should be preceded by a minus sign.

Page 23: In lines 2, 3, and 4, the equations for δ_6 , δ_7 , and δ_8 should be corrected as follows:

$$\delta_6 = \frac{M^2}{48\beta_\sigma^{2/7/2}} \left[4M^2(4\sigma + 5) - \sigma(\sigma - 3) \right]$$

$$\delta_7 = \frac{M^2}{48\beta_\sigma^{2/7/2}} \left[4M^2(2\sigma + 5) + 3\sigma \right]$$

$$\delta_8 = \frac{-M^4\lambda}{48\sigma^{7/2}} (2\sigma + 5)$$

Page 23, next to last line: The superscript " v_2 " should be the superscript " $1/2$."

Page 25, line 4: The equation for H_{03}^1 should be corrected as follows:

$$H_{03}^1 = \frac{1}{4\lambda} F_{04}^1 - \frac{16\beta^2\sigma^{1/2}v^2}{3} \left[\lambda^3 \mu_1^2 + \lambda v^2 (2\sigma + 11) \right] \sqrt{\mu_1^2 - \beta^2 v^2}$$

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SUMMARY

Expressions for the velocity potential and the lift and moment are derived for thin, oscillating, "arrowhead" type wings with supersonic edges moving at a constant supersonic speed. The triangular wing and any one of the series of wings obtained from it by cutting its rearward part so that the flow normal to the resulting trailing edges remains supersonic are included in the term arrowhead type. Explicit results for the lift and moment are given for the wings undergoing harmonic pitching and vertical oscillations.

Closed expressions for the velocity potential, section force and moment coefficients for any arrowhead wing, and total force and moment coefficients for only the triangular or delta wing are developed explicitly to the third power of the frequency of oscillation. These expressions may be sufficient for most practical applications.

The wings considered are found to exhibit the possibility of negative damping in torsion for certain ranges of Mach number and axis-of-rotation position. A figure showing the ranges of these parameters for triangular wings is given herein.

INTRODUCTION

In high-speed-aircraft design, a knowledge of the air forces and moments that act on various types of oscillating-wing plan forms is often required. Such knowledge is useful in considering a number of instability problems, among which are wing flutter and low-frequency instability of the aircraft involving control-surface deflections. The main source of theoretical information has been the solution of the linearized differential equation for compressible flow.

The problem of finding the air forces on an oscillating or steady thin wing in supersonic flow is usually formulated in terms of the disturbance velocity potential for the wing. The disturbance velocity

potential is constructed by superposing solutions of the linearized differential equation, corresponding to sources, doublets, or higher-order singularities, in such a way as to satisfy the boundary conditions at the wing surface and at infinity. The linearized differential equation and its boundary conditions constitute the boundary-value problem for the velocity potential for the wing.

The present paper is concerned with the boundary-value problem for the velocity potential for an apex-forward, thin, flat, oscillating, triangular or related wing with supersonic edges moving at a constant supersonic speed. According to reference 1, this problem may be classified as "purely supersonic," since the upper and lower surfaces of the wing can be regarded as acting independently of one another, and is satisfied by a surface distribution of sources with local strength proportional to the local prescribed normal velocity at the wing surface. Although reference 1 does not explicitly treat the oscillating triangular wing, its general solution must be regarded as giving the velocity potential for this wing in integral form. The purpose of the present paper is then to obtain an integrated form for the velocity potential, from the integral-form solution of reference 1, for the particular case of the oscillating triangular or related wing.

In a strict sense this integrated form applies to a semi-infinite triangular plan form. But, since the wing wake is of no concern in the present development, the velocity potential may be considered to apply to any one of the series of wings obtained from a triangular wing with supersonic edges by cutting the rearward part of this wing in such a manner that the flow normal to the resulting trailing edges remains supersonic. All these wings, including the triangular wing, are henceforth referred to in general as arrowhead wings.

There are several other papers closely associated with the subject being considered. In references 2 and 3 the treatment of oscillating, finite, swept wings involved an integral form for the velocity potential quite similar to that for the oscillating arrowhead wing. In references 4 and 5, more directly, expressions for the total lift and moment were given for the particular oscillating arrowhead wing with supersonic edges known as the "wide" delta wing.

The present paper effectively includes the total forces and moments for the wide delta given in references 4 and 5. In addition the present paper gives the forces and moments on any streamwise wing section for not only the wide delta but also for the more general arrowhead wing. The section rather than the total forces and moments are desirable, for example, in a strip flutter analysis that includes the wing-flutter mode shape.

The velocity potential for the arrowhead wing does not appear to be obtainable in terms of elementary functions (see references 1 and 4).

The solution of this type of problem, therefore, is approximated by various kinds of expansion. Thus in reference 4 the velocity potential was expanded in a series of Bessel functions; whereas in reference 6, in the treatment of the rectangular wing, an expansion in powers of the frequency of oscillation was employed. The frequency-expansion method is utilized in this paper and only the first few terms of the velocity potential and subsequent forces and moments are obtained. The first few terms are considered adequate for a large class of practical applications. In the derivation of the velocity potential and the forces and moments, the wings are assumed to be undergoing harmonic oscillations in pitch and vertical translation.

SYMBOLS

ϕ	disturbance-velocity potential
x, y, z	rectangular Cartesian coordinates
U	velocity of main stream
a	velocity of sound in main stream
t	time
$w(x, y, t)$	normal velocity at surface of wing at point (x, y)
Z_m	function defining vertical displacement of point (x, y) of wing
x_0	abscissa of axis of rotation of wing section as shown in figure 1
ρ	air density in main stream
h	vertical displacement of axis of rotation
h_0	maximum amplitude of vertical displacement of axis of rotation, positive downward
α	angle of attack
α_0	maximum amplitude of angular displacement about axis of rotation, positive leading edge up
$\dot{h}, \dot{\alpha}$	time derivatives of h and α , respectively

ω circular frequency of oscillation

M free-stream Mach number (U/a)

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$$\beta = \sqrt{M^2 - 1}$$

ξ, η rectangular Cartesian coordinates used to represent location of sources in xy-plane

$$\bar{\omega} = \frac{M^2 \omega}{U \beta^2}$$

a_n, b_m functions of $\bar{\omega}$, x , and M defined in equation (11)

λ slope of leading edge of wing as shown in figure 1

$$\vartheta = \lim_{\beta \lambda \rightarrow 1} \phi$$

u, v characteristic coordinates defined in appendix A

$$\gamma = \frac{1 - \beta \lambda}{1 + \beta \lambda}$$

$$\sigma = \beta^2 \lambda^2 - 1$$

b semichord of midspan wing section

μ, ν, μ_0 nondimensional coordinates $\left(\mu = \frac{x}{2b}, \quad \nu = \frac{y}{2b}, \quad \mu_0 = \frac{x_0}{2b} \right)$

k reduced frequency ($b\omega/U$)

Δp local-surface pressure difference

P section force per unit span on wing strip parallel to main stream, positive downward

M_α section moment per unit span on wing strip parallel to main stream, taken about axis of rotation x_0 , positive leading edge up

L_i, M_i components of section force and moment coefficients, respectively, defined in equation (B1); $i = 1, 2, 3, 4$

\bar{P} total force on delta wing, positive downward

\bar{M}_α total moment on delta wing about axis of rotation $x = x_0$, positive leading edge up

\bar{L}_i, \bar{M}_i	components of total force and moment coefficients, respectively, defined in equation (B4); $i = 1, 2, 3, 4$
$\left. \begin{matrix} E_{mn}, F_{mn}, \\ G_{mn}, H_{mn} \end{matrix} \right\}$	functions of μ_1 , β , and v defined in equation (B2)
$\left. \begin{matrix} \alpha_i, \beta_i, \\ \gamma_i, \delta_i \end{matrix} \right\}$	functions of M , λ , and σ defined in equation (B3)
l	distance to point at which extended trailing edge of plan form D, figure 3, intersects x-axis
s	semispan of wing

FORMULATION OF PROBLEM

Consider uniform supersonic flow past a thin, flat, arrowhead type wing as shown in figure 1 (with its leading edges outside the Mach cone generated by the nose of the wing). The wing is referred to a coordinate system fixed in space and is assumed to be creating small disturbances in the main stream flowing past. Then, if the undisturbed stream velocity is in the direction of the positive x-axis and is of magnitude U , the equation satisfied by the disturbance velocity potential for the wing is

$$\frac{1}{a^2} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \phi = \nabla^2 \phi \quad (1)$$

The wing is assumed to be performing oscillations of small amplitude about its own undisturbed position, the plane $Z = 0$. The boundary condition to be satisfied at the wing surface is then

$$\left(\frac{\partial \phi}{\partial z} \right)_{z \rightarrow 0} = w(x, y, t) = U \frac{\partial Z_m}{\partial x} + \frac{\partial Z_m}{\partial t} \quad (2)$$

where Z_m is the vertical displacement of a point (x, y) of the wing. Note that this boundary condition is evaluated, in accordance with small-disturbance linearized theory, at the xy-projection of the wing rather than at the wing itself. The additional boundary conditions, that only the wing can support a pressure difference and that the sources of disturbance must not be felt ahead of their respective Mach cones, are automatically satisfied by the type of source synthesis to be employed in the solutions.

After the boundary-value problem is solved for the velocity potential ϕ , the pressure on either wing surface may be found by means of a linearized form of Bernoulli's equation

$$p = -\rho \left(\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} \right) \quad (3)$$

where ρ is the density in the undisturbed stream.

For the present problem of a wing performing small harmonic torsional oscillations of amplitude α_0 about some spanwise axis x_0 and small harmonic vertical translations of amplitude h_0 (see fig. 1(b)), the equation for Z_m is

$$Z_m = h + (x - x_0)\alpha = \left[h_0 + (x - x_0)\alpha_0 \right] e^{i\omega t} \quad (4)$$

where ω is the frequency of oscillation. Substitution of equation (4) into equation (2) gives

$$w(x, y, t) = \dot{h} + U\alpha + (x - x_0)\dot{\alpha} \quad (5)$$

Since equation (1) is linear, the velocity potential satisfying equations (1) and (5) may be regarded as the sum of three potentials, each being associated with one of the terms on the right-hand side of equation (5). Thus the potential is obtained in the form

$$\phi = \phi_h + \phi_\alpha + \phi_{\dot{\alpha}} \quad (6)$$

Thickness effects are considered negligible and, as a result, the velocity potentials in equation (6) are associated only with lift and are antisymmetrical with respect to the plane $Z = 0$. Thus only one surface of the projected wing need be considered. The top surface ($Z = +0$) is chosen in this analysis and, since the antisymmetrical potential is simply opposite in sign on the bottom surface, the pressure difference supported by the wing is obtained by means of equation (3) as

$$\Delta p = -2\rho \left(\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} \right) \quad (7)$$

EQUATIONS FOR VELOCITY POTENTIAL

Integral Form

The boundary-value problem for the arrowhead wing with supersonic edges, discussed previously, is similar to that for the wing of infinite span treated in reference 1. The problem in reference 1 is shown to be satisfied by a distribution or superposition of sources over the upper wing surface. Thus the velocity potential for the arrowhead wing, irrespective of the type of time variation, is

$$\phi(x, y, +0, t) = - \frac{1}{2\pi} \iint_S w(\xi, \eta) \frac{w(t - \tau_1) + w(t - \tau_2)}{R} d\eta d\xi \quad (8)$$

where $w(\xi, \eta)$ represents the surface variation of source strength and

$$\tau_{1,2} = \frac{M(x - \xi) \mp R}{a\beta^2}$$

$$R = \sqrt{(x - \xi)^2 - \beta^2(y - \eta)^2}$$

The region of integration S , in equation (8), is the part of the wing (in $\xi\eta$ -plane) cut out by the reflected Mach cone opening upstream of the point $(x, y, +0)$ as shown by the shaded area in figure 2.

For the present problem of the wing performing small harmonic torsional and vertical oscillations, the time variation of source strength is given by

$$w(t) = e^{i\omega t}$$

Thus, as shown in reference 1, equation (8) takes the form

$$\phi(x, y, +0, t) = - \frac{e^{i\omega t}}{\pi} \iint_S \frac{w(\xi, \eta)}{R} e^{-i(x-\xi)\bar{\omega}} \cos\left(\bar{\omega} \frac{R}{M}\right) d\eta d\xi \quad (9)$$

where

$$\bar{\omega} = \frac{\omega M}{a\beta^2} = \frac{M^2 \omega}{U\beta^2}$$

and the values of $w(\xi, \eta)$ for the harmonic case from equation (5) are:

For h ,

$$w(\xi, \eta) = i\omega h_0 = \frac{iU\beta^2 \bar{\omega}}{M^2} h_0$$

For $U\alpha$,

$$w(\xi, \eta) = U\alpha_0$$

For $\dot{\alpha}(x - x_0)$,

$$w(\xi, \eta) = \frac{iU\beta^2 \bar{\omega}}{M^2} (\xi - x_0) \alpha_0$$

For certain parts of the region of the wing (shaded in fig. 2) to which equation (9) applies, the integration can be carried out as in the case of the infinite wing considered in reference 1. However, the entire potential ϕ does not appear to be obtainable in terms of known functions. Previous mention of this point was made in the introduction. In order to give consistent treatment to the entire region of integration, the integrand of equation (9) may be expanded in Maclaurin's series with respect to $\bar{\omega}$, with the result that each term of the expansion can be readily integrated. Thus equation (9) is used in the following form:

$$\phi(x, y, +0, t) = -\frac{e^{i\omega t}}{\pi} \iint_S \frac{w(\xi, \eta)}{R} \sum_{m=0}^{\infty} \bar{\omega}^m \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{(-i)^m (x - \xi)^{m-2n}}{(2n)!(m-2n)!} \left(\frac{R}{M}\right)^{2n} d\eta d\xi \quad (10)$$

where $\left[\frac{m}{2}\right]$ denotes the integral part of $m/2$.

Integrated Form

The velocity potential to the third power of $\bar{\omega}$ is considered sufficient for a large number of practical applications. Thus equation (10) to the third power of $\bar{\omega}$, with any $\bar{\omega}$ that may come from $w(\xi, \eta, t)$ temporarily neglected, is

$$\begin{aligned} \phi(x, y, +0, t) = -\frac{e^{i\omega t}}{\pi} \iint_S w(\xi, \eta) & \left(a_0 \frac{1}{R} + a_1 \frac{\xi}{R} + a_2 \frac{\xi^2}{R} + \right. \\ & \left. a_3 \frac{\xi^3}{R} + b_0 R + b_1 \xi R \right) d\eta d\xi \end{aligned} \quad (11)$$

where

$$a_0 = 1 - i\bar{\omega}x - \frac{\bar{\omega}^2}{2} x^2 + i \frac{\bar{\omega}^3}{6} x^3$$

$$a_1 = i\bar{\omega} + \bar{\omega}^2 x - i \frac{\bar{\omega}^3}{2} x^2$$

$$a_2 = -\frac{\bar{\omega}^2}{2} + i \frac{\bar{\omega}^3}{2} x$$

$$a_3 = -i \frac{\bar{\omega}^3}{6}$$

$$b_0 = -\frac{\bar{\omega}^2}{2M^2} + i \frac{\bar{\omega}^3}{2M^2} x$$

$$b_1 = -i \frac{\bar{\omega}^3}{2M^2}$$

The question may be raised at this point as to whether equation (11) still represents a solution of the boundary-value problem under discussion. With regard to this question, it can be shown that, when all the terms involving $\bar{\omega}$ up to a given power are taken into account, the differential equation (1) is satisfied to this power. The boundary condition of tangential flow, equation (5), is satisfied exactly, regardless of the order of $\bar{\omega}$ considered.

Putting the values of $w(\xi, \eta)$ given in equation (9) into equation (11) and performing the indicated surface integration (over the shaded region in fig. 2) yields the following forms for the velocity potentials in equation (6):

$$\left. \begin{aligned} \phi_h &= -\frac{h}{\pi} \left[A \sqrt{x^2 - \beta^2 y^2} + B_1 \cos^{-1} \frac{x + \beta^2 \lambda y}{\beta(\lambda x + y)} + B_2 \cos^{-1} \frac{x - \beta^2 \lambda y}{\beta(\lambda x - y)} \right] \\ &= -\frac{h}{\pi} F_1(x, y) \\ \phi_\alpha &= -\frac{U\alpha}{\pi} F_1(x, y) \\ \phi_\alpha^* &= -\frac{\dot{\alpha}}{\pi} \left[C \sqrt{x^2 - \beta^2 y^2} + D_1 \cos^{-1} \frac{x + \beta^2 \lambda y}{\beta(\lambda x + y)} + D_2 \cos^{-1} \frac{x - \beta^2 \lambda y}{\beta(\lambda x - y)} - \right. \\ &\quad \left. x_0 F_1(x, y) \right] \end{aligned} \right\} \quad (12)$$

The coefficients A , B_1 , B_2 , C , D_1 , and D_2 are given in appendix A. In order to perform the integrations that led to the results represented by equation (12), the use of characteristic coordinates (Mach lines in fig. 2) was found convenient. Reasons for using these coordinates are given in appendix A, where a brief account of the derivation of equation (12) is presented.

In the limiting case where the Mach lines coincide with the leading edges of the wing, that is, when $\beta\lambda = 1$, equations (12) take the following form:

$$\left. \begin{aligned} \varphi_h &= -\frac{h}{\pi} A_1 \sqrt{x^2 - \beta^2 y^2} \\ \varphi_\alpha &= -\frac{U\alpha}{\pi} A_1 \sqrt{x^2 - \beta^2 y^2} \\ \varphi_{\dot{\alpha}} &= -\frac{\dot{\alpha}}{\pi} (C_1 - x_0 A_1) \sqrt{x^2 - \beta^2 y^2} \end{aligned} \right\} \quad (13)$$

The coefficients A_1 and C_1 and the derivation of equation (13) are also given in appendix A.

FORCES AND MOMENTS

The preceding result for the velocity potential, equation (12), is now used to obtain the forces and moments on any one of a series of wings generally referred to herein as arrowhead type. As stated in the introduction, the wings being considered are those that may be formed from the delta or triangular wing by cutting the rearward part of this wing in such a manner that the resulting trailing edges lie ahead of the Mach cones emanating from their foremost points. Sketches of several wing plan forms having the aforementioned characteristics are shown in figure 3.

The forces and moments acting on a wing section, such as section y of figure 1, are derived in general form, that is, in the form applicable to any of the aforementioned wings. These forces and moments are useful, for example, in a strip flutter analysis. Total forces and moments are also derived, but only for the delta wing. The damping part of the total moment due to α_0 is used in a study of the possible loss of aerodynamic damping (loss indicates that single-degree torsional flutter is possible) of an oscillating delta wing, which infers the same possibility with regard to the other wings considered.

In deriving the expressions for the section and total force and moment coefficients, it is convenient to employ the variables μ , v , and μ_0 as the nondimensional quantities, obtained by dividing the variables x , y , and x_0 by the chord $2b$ of the midspan section of the wing, and to introduce the reduced frequency $k = \frac{b\omega}{U}$.

Section Forces and Moments

The local pressure difference between the upper and lower surfaces of the wing is given by means of equation (7) as

$$\Delta p = -2\rho \left(\frac{\partial \phi}{\partial t} + \frac{U}{2b} \frac{\partial \phi}{\partial \mu} \right) \quad (14)$$

The expression for the section force, positive downward, is therefore

$$P = -2b \int_{v/\lambda}^{\mu_1} \Delta p \, d\mu \quad (15)$$

and the section moment, positive nose up, about the arbitrary axis of rotation $x = x_0$ is

$$M_\alpha = -4b^2 \int_{v/\lambda}^{\mu_1} (\mu - \mu_0) \Delta p \, d\mu \quad (16)$$

The limit of integration μ_1 in equations (15) and (16) has the following values with respect to the different plan forms shown in figure 3:

For plan form A,

$$\mu_1 = 1$$

For plan form B,

$$\mu_1 = 1 - \frac{v}{m}$$

For plan form C,

$$\mu_1 = 1 + \frac{v}{m}$$

For plan form D,

$$\mu_1 = 1 + \frac{v}{m_1} \quad \text{for } 0 \leq v \leq \frac{m_1 m_2}{m_1 + m_2} \left(\frac{l}{2b} - 1 \right)$$

$$\mu_1 = \frac{l}{2b} - \frac{v}{m_2} \quad \text{for } \frac{m_1 m_2}{m_1 + m_2} \left(\frac{l}{2b} - 1 \right) \leq v \leq \frac{s}{2b}$$

(17)

After equation (12) is substituted into equations (15) and (16) and the indicated integrations are performed, the results can be written as

$$P = -4\rho b U^2 k^2 e^{i\omega t} \left[\frac{h_0}{b} (L_1 + iL_2) + \alpha_0 (L_3 + iL_4) \right] \quad (18)$$

and

$$M_\alpha = -4\rho b U^2 k^2 e^{i\omega t} \left[\frac{h_0}{b} (M_1 + iM_2) + \alpha_0 (M_3 + iM_4) \right] \quad (19)$$

In equations (18) and (19) the reduced frequency k , on which the L_i 's and M_i 's are dependent, is related to ω and $\bar{\omega}$ in the following manner:

$$k = \frac{b\omega}{U} = \frac{b\beta^2}{M^2} \bar{\omega}$$

A brief account of the derivation of and expressions for the force and moment coefficients L_i and M_i (where $i = 1, 2, 3, 4$) appearing in equations (18) and (19) is given in appendix B.

In equations (18) and (19) the coefficients $L_1 + iL_2$, $M_1 + iM_2$ and $L_3 + iL_4$, $M_3 + iM_4$ are the section lift and moment coefficients associated with vertical translation and rotation, respectively. The real component of $L_1 + iL_2$, for example, is in phase with the displacement h_0 and the imaginary part is 90° out of phase with this displacement. Similar interpretations can be given to the remaining components. The imaginary part of each of the coefficients is proportional to the aerodynamic damping force or moment associated with the respective motion.

Total Forces and Moments for Delta Wing

The expression for the total force, positive downward, on plan form A of figure 3 is

$$\bar{P} = -8b^2 \int_0^\lambda \int_{v/\lambda}^1 \Delta p \, d\mu \, dv \quad (20)$$

The total moment, positive nose up, about the arbitrary axis of rotation $x = x_0$ is

$$\bar{M}_\alpha = -16b^3 \int_0^\lambda \int_{v/\lambda}^1 \Delta p (x - x_0) d\mu dv \quad (21)$$

After equation (12) is substituted into equations (20) and (21) and the indicated integrations are performed, the results can be written as (using barred letters throughout to designate total forces and moments)

$$\bar{P} = -8\rho b^2 U^2 k^2 e^{i\omega t} \left[\frac{h_0}{b} (\bar{L}_1 + i\bar{L}_2) + \alpha_0 (\bar{L}_3 + i\bar{L}_4) \right] \quad (22)$$

and

$$\bar{M}_\alpha = -8\rho b^3 U^2 k^2 e^{i\omega t} \left[\frac{h_0}{b} (\bar{M}_1 + i\bar{M}_2) + \alpha_0 (\bar{M}_3 + i\bar{M}_4) \right] \quad (23)$$

The derivation of the total force and moment coefficients \bar{L}_i and \bar{M}_i ($i = 1, 2, 3, 4$) is also given in appendix B.

SOME CALCULATIONS AND DISCUSSION

It may be of interest to the reader to examine the spanwise distributions of the various components of the section lift and moment coefficients for a particular case. Thus, the components of equations (18) and (19), given more fully in equation (B1), have been evaluated at different spanwise positions y for plan form A of figure 3 for the following set of conditions: $\lambda = \sqrt{3}$, $\mu_0 = 0.5$, $M^2 = 1.75$, and $k = 0.04$. These sample results are plotted as functions of the span position in figure 4. The spanwise variations of the section-force components are shown in figure 4(a) and the corresponding variations of the section-moment coefficients in figure 4(b).

The components of the total force and moment coefficients (given by equation (B4)) have also been evaluated for the previous set of conditions. The results, after being referred to the nondimensional span of the wing 2λ , are represented by dashed lines in figure 4. As should be the case, the areas under the dashed lines are equal to those under the respective section-component curves.

As a result of the linear dependence of the total force and moment coefficients on the semiapex angle of the wing, as may be noted in equation (B4), the average ordinates of the distributions given in

figure 4 will not change if the apex angle is changed. Only the shapes of the section force or moment distributions will be affected by such a variation in angle.

In the example plotted, the total component of moment coefficient \bar{M}_4 is negative. This term would therefore not contribute to the aerodynamic damping but would act as a source of energy for the oscillating system. This result is significant since it indicates the possibility of single-degree torsional flutter.

The wing plan forms discussed in this paper exhibit the possibility of loss of aerodynamic damping in torsion for certain ranges of Mach number M and axis-of-rotation location μ_0 . This possibility is indicated, as mentioned in the previous paragraph, by the negative value of the torsional damping-moment component \bar{M}_4 in the example given. Since a wing oscillates as a rigid body at a fairly low frequency of oscillation, the main results of the loss-of-damping phenomenon can be obtained by maintaining \bar{M}_4 to the order of $1/k$ only. Thus, there is obtained for plan form A of figure 3 the following result from equation (B4):

$$\bar{M}_4 = \frac{\lambda}{3\beta^3 k} \left[12(M^2 - 1)\mu_0^2 - 4\mu_0(4M^2 - 5) + 3(2M^2 - 3) \right] \quad (24)$$

In general, torsional stability depends on the sign and magnitude of \bar{M}_4 . Positive values of \bar{M}_4 indicate stability and negative values indicate possible instability. The borderline case is thus given by $\bar{M}_4 = 0$.

The ranges of values of Mach number M and axis of rotation μ_0 , for which \bar{M}_4 in equation (24) vanishes, are shown in figure 5. Only one curve is obtained for all delta wings with supersonic edges, that is, $\beta\lambda \geq 1$, because \bar{M}_4 is dependent on λ for magnitude only, as may be noted in equation (24). The region inside the curve in figure 5 is the region of possible instability. For convenience, a second ordinate is given in figure 5 showing the values of λ below which the leading edges become subsonic. With the aid of this second ordinate it may be seen that a delta wing with a semivertex angle of 45° is the narrowest triangular wing that may show a torsional instability and yet retain the characteristic $\beta\lambda \geq 1$ because the uppermost part of the

curve in the figure corresponds to a value of $\lambda = 1$ when $\beta\lambda = 1$. For a narrower wing to have at least sonic edges, it must be moving at a Mach number which would place it in the stable region of figure 5.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
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APPENDIX A

COEFFICIENTS OF VELOCITY-POTENTIAL EQUATIONS

A derivation of equations (12) and (13) is given briefly here. The surface integral represented by equation (11) when written in terms of the characteristic coordinates u, v becomes

$$\phi = -\frac{e}{\pi} \frac{i\omega t}{\pi} \left[\sum_{n=0}^3 \frac{\beta^n}{M^{n+1}} \int_{v_1}^{v_2} \int_{u_1}^{u_2} w(u, v) a_n \frac{(u+v)^n du dv}{\sqrt{(u_0-u)(v_0-v)}} + \right. \\ \left. \sum_{m=0}^1 \frac{4\beta^{m+2}}{M^{m+3}} \int_{v_1}^{v_2} \int_{u_1}^{u_2} w(u, v) b_m (u+v)^m \sqrt{(u_0-u)(v_0-v)} du dv \right] \quad (A1)$$

where

$$u = \frac{M}{2\beta} (\xi - \beta\eta)$$

$$u_0 = \frac{M}{2\beta} (x - \beta y)$$

$$v = \frac{M}{2\beta} (\xi + \beta\eta)$$

$$v_0 = \frac{M}{2\beta} (x + \beta y)$$

The limits of integration in equation (A1), from the characteristic coordinates previously defined and from figure 2, are:

Region	u_1	u_2	v_1	v_2
I	0	u_0	0	v_0
II	$\frac{1}{\gamma} v$	u_0	γu_0	0
III	γv	0	0	v_0

(A2)

where

$$\gamma = \frac{1 - \beta\lambda}{1 + \beta\lambda}$$

Putting the appropriate values of $w(u,v)$ and the limits of integration given in equation (A2) into equation (A1) and performing the indicated integrations, and then reverting to xy -coordinates, yields the potentials in the forms given in equation (12). Retaining only those terms associated with h or α that involve \bar{w} up to the third power yields the following values for the coefficients in equation (12):

$$\begin{aligned} A &= i \frac{\lambda x \bar{w}}{\sigma} + \frac{\lambda x^2 \bar{w}^2}{6M^2 \sigma^2} \left[M^2 (4\sigma + 3) + \sigma \right] + \frac{\lambda \beta^2 y^2 \bar{w}^2}{6M^2 \sigma^2} (3M^2 + \sigma) - \\ &\quad i \frac{\lambda x^3 \bar{w}^3}{72M^2 \sigma^3} \left[M^2 (18\sigma^2 + 31\sigma + 15) + 3\sigma(5\sigma + 3) \right] - \\ &\quad i \frac{\lambda \beta^2 x y^2 \bar{w}^3}{72M^2 \sigma^3} \left[M^2 (38\sigma + 45) + 3\sigma(4\sigma + 9) \right] \\ B_1(x,y) &= \frac{1}{\sigma^{1/2}} (\lambda x + y) - i \frac{\lambda \beta^2 \bar{w}}{2\sigma^{3/2}} (\lambda x + y)^2 - \frac{\beta^2 \bar{w}^2}{12M^2 \sigma^{5/2}} \left[M^2 (2\sigma + 3) + \right. \\ &\quad \left. \sigma \right] (\lambda x + y)^3 + i \frac{\lambda \beta^4 \bar{w}^3}{48M^2 \sigma^{7/2}} \left[M^2 (2\sigma + 5) + 3\sigma \right] (\lambda x + y)^4 \\ B_2(x,y) &= B_1(x,-y) \\ C &= \frac{\lambda x}{\sigma} - i \frac{\lambda x^2 \bar{w}}{3\sigma^2} (\sigma + 3) - i \frac{\lambda \beta^2 y^2 \bar{w}}{\sigma^2} - \frac{\lambda x^3 \bar{w}^2}{24M^2 \sigma^3} \left[M^2 (2\sigma^2 + 19\sigma + 15) + \right. \\ &\quad \left. \sigma(\sigma + 3) \right] - \frac{\lambda \beta^2 x y^2 \bar{w}^2}{24M^2 \sigma^3} \left[M^2 (26\sigma + 45) + 9\sigma \right] \end{aligned} \quad (A3)$$

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$$\begin{aligned}
 D_1(x, y) = & -\frac{x}{\sigma^{3/2}}(\lambda x + y) - \frac{\lambda \beta^2 y}{\sigma^{3/2}}(\lambda x + y) + \frac{\lambda \beta^2}{2\sigma^{3/2}}(\lambda x + y)^2 - \\
 & i \frac{\lambda \beta^2 x \bar{w}}{2\sigma^{3/2}}(\lambda x + y)^2 + i \frac{\beta^2 \bar{w}}{6\sigma^{5/2}}(2\sigma + 3)(\lambda x + y)^3 + \\
 & \frac{\beta^2 x \bar{w}^2}{48M^2 \sigma^{7/2}} \left[4M^2(4\sigma + 5) - \sigma(\sigma - 3) \right] (\lambda x + y)^3 + \\
 & \frac{\lambda \beta^4 y \bar{w}^2}{48M^2 \sigma^{7/2}} \left[4M^2(2\sigma + 5) + 3\sigma \right] (\lambda x + y)^3 - \\
 & \frac{\lambda \beta^4 \bar{w}^2}{48\sigma^{7/2}}(2\sigma + 5)(\lambda x + y)^4
 \end{aligned}
 \quad (A3)$$

$$D_2(x, y) = D_1(x, -y)$$

where

$$\sigma = \beta^2 \lambda^2 - 1$$

For the case in which the velocity components normal to the leading edges are sonic or in which the Mach lines coincide with these edges (that is, when $\beta\lambda = 1$, $\gamma = 0$, or $\sigma = 0$), it may be noted from equation (A2) and figure 2 that regions II and III no longer exist. Therefore, the potential in this case is determined by integration over region I only. The coefficients of equation (13) are thus found to be:

$$\begin{aligned}
 A_1 = & \frac{2}{\beta} \left\{ 1 - i \frac{x \bar{w}}{3} - \frac{\bar{w}^2}{90M^2} \left[5(x^2 - \beta^2 y^2) + M^2(7x^2 + 2\beta^2 y^2) \right] + \right. \\
 & \left. i \frac{x \bar{w}^3}{210M^2} \left[7(x^2 - \beta^2 y^2) + M^2(3x^2 + 2\beta^2 y^2) \right] \right\} \\
 C_1 = & \frac{2}{\beta} \left\{ \frac{2x}{3} - i \frac{2\bar{w}}{45} (4x^2 - \beta^2 y^2) - \frac{x \bar{w}^2}{315M^2} \left[7(x^2 - \beta^2 y^2) + M^2(11x^2 - 2\beta^2 y^2) \right] \right\}
 \end{aligned}
 \quad (A4)$$

The use of characteristic coordinates leads to very simple expressions for the limits of integration given in equation (A2). In addition, their use simplifies the derivation of equation (A4).

APPENDIX B

FORCE AND MOMENT COEFFICIENTS

Section Force and Moment Coefficients

The coefficients L_i and M_i (where $i = 1, 2, 3, 4$) are obtained by substituting equation (12) into equations (15) and (16) and grouping the results in the form given in equations (18) and (19). These coefficients are therefore defined as follows:

$$\left. \begin{aligned} L_1 + iL_2 &= L_1' + iL_2' \\ L_3 + iL_4 &= L_3' + iL_4' - \left(\frac{i}{k} + 2\mu_0\right)(L_1' + iL_2') \\ M_1 + iM_2 &= M_1' + iM_2' - 2\mu_0(L_1' + iL_2') \\ M_3 + iM_4 &= M_3' + iM_4' - 2\mu_0(L_3' + iL_4') - \left(\frac{i}{k} + 2\mu_0\right)(M_1' + iM_2') \end{aligned} \right\} \quad (B1)$$

where

$$L_1' + iL_2' = -\frac{1}{2\pi} \left[2 \int_{v/\lambda}^{\mu_1} F_1(2\mu, 2v) d\mu - \frac{i}{k} F_1(2\mu_1, 2v) \right]$$

$$L_3' + iL_4' = -\frac{1}{2\pi} \left[2 \int_{v/\lambda}^{\mu_1} F_2(2\mu, 2v) d\mu - \frac{i}{k} F_2(2\mu_1, 2v) \right]$$

$$M_1' + iM_2' = -\frac{1}{2\pi} \left[4 \int_{v/\lambda}^{\mu_1} F_1(2\mu, 2v) \mu d\mu - \frac{i}{k} 2\mu_1 F_1(2\mu_1, 2v) \right]$$

$$M_3' + iM_4' = -\frac{1}{2\pi} \left[4 \int_{v/\lambda}^{\mu_1} F_2(2\mu, 2v) \mu d\mu - \frac{i}{k} 2\mu_1 F_2(2\mu_1, 2v) \right]$$

In equations (B1) the quantity $F_1(2\mu, 2v)$ is obtained from the quantity $F_1(x, y)$, defined in connection with equation (12), by replacing x

by 2μ , y by $2v$, and regarding the $\bar{\omega}$ associated with the coefficients A , B_1 , and B_2 in $F_1(x,y)$ as $\bar{\omega}'$ where

$$\bar{\omega}' = \frac{M^2 k}{\beta^2}$$

The quantity $F_2(2\mu, 2v)$ is obtained in a similar manner from the expression

$$F_2(x,y) = C \sqrt{x^2 - \beta^2 y^2} + D_1 \cos^{-1} \frac{x + \beta^2 \lambda y}{\beta(\lambda x + y)} + D_2 \cos^{-1} \frac{x - \beta^2 \lambda y}{\beta(\lambda x - y)}$$

the right-hand side of which is recognized as part of ϕ_α in equation (12).

Consider now the following general definitions, particular forms of which are contained in L_i and M_i ($i = 1, 2, 3, 4$):

$$\left. \begin{aligned} E_{mn} &= 2^{m+n+1} v^m \mu_1^n \sqrt{\mu_1^2 - \beta^2 v^2} \\ F_{mn}^e &= 2^{e+m+n} \left[(\beta^2 \lambda v)^e \mu_1^m (\lambda \mu_1 + v)^n \cos^{-1} \frac{\mu_1 + \beta^2 \lambda v}{\beta(\lambda \mu_1 + v)} + \right. \\ &\quad \left. (-\beta^2 \lambda v)^e \mu_1^m (\lambda \mu_1 - v)^n \cos^{-1} \frac{\mu_1 - \beta^2 \lambda v}{\beta(\lambda \mu_1 - v)} \right] \\ G_{mn} &= 2^{m+n+2} v^m \int_{v/\lambda}^{\mu_1} \mu^n \sqrt{\mu^2 - \beta^2 v^2} d\mu \\ H_{mn}^e &= 2^{e+m+n+1} \left[(\beta^2 \lambda v)^e \int_{v/\lambda}^{\mu_1} \mu^m (\lambda \mu + v)^n \cos^{-1} \frac{\mu + \beta^2 \lambda v}{\beta(\lambda \mu + v)} d\mu + \right. \\ &\quad \left. (-\beta^2 \lambda v)^e \int_{v/\lambda}^{\mu_1} \mu^m (\lambda \mu - v)^n \cos^{-1} \frac{\mu - \beta^2 \lambda v}{\beta(\lambda \mu - v)} d\mu \right] \end{aligned} \right\} \quad (B2)$$

With the aid of equation (B2), the L_i' and M_i' ($i = 1, 2, 3, 4$) of equation (B1) may be written as

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$$-2\pi L_1' = (\alpha_1 E_{01} + \beta_1 H_{01}^0 + \beta_2 F_{02}^0) + (\alpha_2 G_{02} + \alpha_3 G_{20} + \alpha_4 E_{03} + \alpha_5 E_{21} + \beta_3 H_{03}^0 + \beta_4 F_{04}^0) k^2$$

$$-2\pi L_2' = -\beta_1 F_{01}^0 \frac{1}{k} + (\alpha_1 G_{01} - \alpha_2 E_{02} - \alpha_3 E_{20} + \beta_2 H_{02}^0 - \beta_3 F_{03}^0) k$$

$$-2\pi L_3' = \gamma_1 G_{01} + \gamma_2 E_{02} + \gamma_3 E_{20} + \delta_1 H_{11}^0 + \delta_2 H_{01}^1 + \delta_3 H_{02}^0 + \delta_4 F_{12}^0 + \delta_5 F_{03}^0$$

$$-2\pi L_4' = -(\gamma_1 E_{01} + \delta_1 F_{11}^0 + \delta_2 F_{01}^1 + \delta_3 F_{02}^0) \frac{1}{k} + (\gamma_2 G_{02} + \gamma_3 G_{20} - \gamma_4 E_{03} - \gamma_5 E_{21} + \delta_4 H_{12}^0 + \delta_5 H_{03}^0 - \delta_6 F_{13}^0 - \delta_7 F_{03}^1 - \delta_8 F_{04}^0) k$$

$$-2\pi M_1' = \left[\alpha_1 (E_{02} - G_{01}) + \beta_1 H_{11}^0 + \beta_2 (F_{12}^0 - H_{02}^0) \right] + \left[\alpha_2 G_{03} + \alpha_3 G_{21} + \alpha_4 (E_{04} - G_{03}) + \alpha_5 (E_{22} - G_{21}) + \beta_3 H_{13}^0 + \beta_4 (F_{14}^0 - H_{04}^0) \right] k^2 \quad (B3)$$

$$-2\pi M_2' = \beta_1 (H_{01}^0 - F_{11}^0) \frac{1}{k} + \left[\alpha_1 G_{02} + \alpha_2 (G_{02} - E_{03}) + \alpha_3 (G_{20} - E_{21}) + \beta_2 H_{12}^0 + \beta_3 (H_{03}^0 - F_{13}^0) \right] k$$

$$-2\pi M_3' = \gamma_1 G_{02} + \gamma_2 (E_{03} - G_{02}) + \gamma_3 (E_{21} - G_{20}) + \delta_1 H_{21}^0 + \delta_2 H_{11}^1 + \delta_3 H_{12}^0 + \delta_4 (F_{22}^0 - H_{12}^0) + \delta_5 (F_{13}^0 - H_{03}^0)$$

$$-2\pi M_4' = \left[\gamma_1 (G_{01} - E_{02}) + \delta_1 (H_{11}^0 - F_{21}^0) + \delta_2 (H_{01}^1 - F_{11}^1) + \delta_3 (H_{02}^0 - F_{12}^0) \right] \frac{1}{k} + \left[\gamma_2 G_{03} + \gamma_3 G_{21} + \gamma_4 (G_{03} - E_{04}) + \gamma_5 (G_{21} - E_{22} + \delta_4 H_{22}^0 + \delta_5 H_{13}^0 + \delta_6 (H_{13}^0 - F_{23}^0) + \delta_7 (H_{03}^1 - F_{13}^1) + \delta_8 (H_{04}^0 - F_{14}^0) \right] k$$

where

$$\alpha_1 = \frac{M^2 \lambda}{\beta^2 \sigma}$$

$$\alpha_2 = \frac{M^2 \lambda}{6\beta^4 \sigma^2} \left[M^2 (4\sigma + 3) + \sigma \right]$$

$$\alpha_3 = \frac{M^2 \lambda}{6\beta^2 \sigma^2} (3M^2 + \sigma)$$

$$\alpha_4 = -\frac{M^4 \lambda}{72\beta^6 \sigma^3} \left[M^2 (18\sigma^2 + 31\sigma + 15) + 3\sigma(5\sigma + 3) \right]$$

$$\alpha_5 = -\frac{M^4 \lambda}{72\beta^4 \sigma^3} \left[M^2 (38\sigma + 45) + 3\sigma(4\sigma + 9) \right]$$

$$\beta_1 = \frac{1}{\sigma^{1/2}}$$

$$\beta_3 = \overset{\text{minus}}{-} \frac{M^2}{12\beta^2 \sigma^{5/2}} \left[M^2 (2\sigma + 3) + \sigma \right]$$

$$\beta_2 = -\frac{M^2 \lambda}{2\sigma^{3/2}}$$

$$\beta_4 = \frac{M^4 \lambda}{48\beta^2 \sigma^{7/2}} \left[M^2 (2\sigma + 5) + 3\sigma \right]$$

$$\gamma_1 = \frac{\lambda}{\sigma}$$

$$\gamma_3 = -\frac{M^2 \lambda}{\sigma^2}$$

$$\gamma_2 = -\frac{M^2 \lambda}{3\beta^2 \sigma^2} (\sigma + 3)$$

$$\gamma_4 = -\frac{M^2 \lambda}{24\beta^4 \sigma^3} \left[M^2 (2\sigma^2 + 19\sigma + 15) + \sigma(\sigma + 3) \right]$$

$$\gamma_5 = -\frac{M^2 \lambda}{24\beta^2 \sigma^3} \left[M^2 (26\sigma + 45) + 9\sigma \right]$$

$$\delta_1 = -\frac{1}{\sigma^{3/2}}$$

$$\delta_5 = \frac{M^2}{6\sigma^{5/2}}(2\sigma + 3)$$

$$\delta_2 = \delta_1$$

$$\delta_6 = \frac{M^2}{48\beta^2\sigma^{7/2}} \left[4M^2(4\sigma + \frac{5}{2}) - \sigma(\sigma - 3) \right]$$

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$$\delta_3 = \frac{\beta^2\lambda}{2\sigma^{3/2}}$$

$$\delta_7 = \frac{M^2}{48\beta^2\sigma^{7/2}} \left[4M^2(2\sigma + \frac{5}{2}) + 3\sigma \right]$$

$$\delta_4 = -\frac{M^2\lambda}{2\sigma^{3/2}}$$

$$\delta_8 = \frac{-M^4\lambda}{48\sigma^{7/2}}(2\sigma + \frac{5}{2})$$

The coefficients in equation (B3), that is, α_i , β_i , γ_i , and δ_i , originate, respectively, from the coefficients A, B_1 or B_2 , C, and D_1 or D_2 defined in appendix A. The quantities E_{mn} and F_{mn} contained in equation (B3) are easily determined from equation (B2). The quantities G_{mn} and H_{mn} , however, are determined by integration, as indicated in equation (B2). The integral results that are required in equation (B3) are:

$$G_{01} = \frac{8}{3}(\mu_1^2 - \beta^2 v^2)^{3/2}$$

$$G_{02} = 2 \left[2\mu_1(\mu_1^2 - \beta^2 v^2)^{3/2} + \beta^2 v^2 \mu_1(\mu_1^2 - \beta^2 v^2)^{1/2} - \beta^4 v^4 \cosh^{-1} \frac{\mu_1}{\beta v} \right]$$

$$G_{03} = \frac{32}{15}(3\mu_1^2 + 2\beta^2 v^2)(\mu_1^2 - \beta^2 v^2)^{1/2}$$

$$G_{20} = 8v^2 \left[\mu_1(\mu_1^2 - \beta^2 v^2)^{1/2} - \beta^2 v^2 \cosh^{-1} \frac{\mu_1}{\beta v} \right]$$

$$G_{21} = \frac{32v^2}{3}(\mu_1^2 - \beta^2 v^2)^{3/2}$$

$$H_{01}^0 = \frac{1}{2\lambda} F_{02}^0 - \frac{4\sigma^{1/2}}{\lambda} v^2 \cosh^{-1} \frac{\mu_1}{\beta v}$$

$$H_{02}^0 = \frac{1}{3\lambda} F_{03}^0 - \frac{32\sigma^{1/2}}{3} v^2 \sqrt{\mu_1^2 - \beta^2 v^2}$$

$$H_{03}^0 = \frac{1}{4\lambda} F_{04}^0 - \frac{4\sigma^{1/2}}{\lambda} \left[3\mu_1 \lambda^2 v^2 \sqrt{\mu_1^2 - \beta^2 v^2} + v^4 (3\sigma + 5) \cosh^{-1} \frac{\mu_1}{\beta v} \right]$$

$$H_{04}^0 = \frac{1}{5\lambda} F_{05}^0 - \frac{256\sigma^{1/2}}{15\lambda} \left[\lambda v^4 (2\sigma + 5) + \lambda^3 \mu_1^2 v^2 \right] \sqrt{\mu_1^2 - \beta^2 v^2}$$

$$H_{11}^0 = -\frac{1}{6\lambda^2} F_{03}^0 + \frac{1}{2\lambda} F_{12}^0 - \frac{8\sigma^{1/2}}{3\lambda} v^2 \sqrt{\mu_1^2 - \beta^2 v^2}$$

$$H_{12}^0 = -\frac{1}{12\lambda^2} F_{04}^0 + \frac{1}{3\lambda} F_{13}^0 - \frac{4\sigma^{1/2}}{3\lambda^2} \left[5\lambda^2 v^2 \mu_1 \sqrt{\mu_1^2 - \beta^2 v^2} + v^4 (5\sigma + 3) \cosh^{-1} \frac{\mu_1}{\beta v} \right]$$

$$H_{13}^0 = -\frac{1}{20\lambda^2} F_{05}^0 + \frac{1}{4\lambda} F_{14}^0 - \frac{16\sigma^{1/2}}{15\lambda^2} \left[(22\sigma + 25) \lambda v^4 + 11\lambda^3 v^2 \mu_1^2 \right] \sqrt{\mu_1^2 - \beta^2 v^2}$$

$$H_{21}^0 = \frac{1}{12\lambda^3} F_{04}^0 - \frac{1}{3\lambda^2} F_{13}^0 + \frac{1}{2\lambda} F_{22}^0 - \frac{4\sigma^{1/2}}{3\lambda^2} \left[\lambda^2 v^2 \mu_1 \sqrt{\mu_1^2 - \beta^2 v^2} + v^4 (\sigma + 3) \cosh^{-1} \frac{\mu_1}{\beta v} \right]$$

$$H_{22}^0 = \frac{1}{30\lambda^3} F_{05}^0 - \frac{1}{6\lambda^2} F_{14}^0 + \frac{1}{3\lambda} F_{23}^0 - \frac{32\sigma^{1/2}}{15\lambda^3} \left[3\lambda^3 v^2 \mu_1^2 + \lambda v^4 (6\sigma + 5) \right] \sqrt{\mu_1^2 - \beta^2 v^2}$$

$$H_{01}^1 = \frac{1}{2\lambda} F_{02}^1 - 8\sigma^{1/2} \beta^2 \lambda v^2 \sqrt{\mu_1^2 - \beta^2 v^2}$$

$$H_{11}^1 = -\frac{\beta^2}{6\lambda} F_{04}^0 + \frac{2\beta^2}{3} F_{13}^0 - \frac{\beta^2 \lambda}{2} F_{22}^0 - \frac{16\beta^2 \sigma^{1/2}}{3\lambda} \left(\lambda^2 v^2 \mu_1 \sqrt{\mu_1^2 - \beta^2 v^2} + v^4 \cosh^{-1} \frac{\mu_1}{\beta v} \right)$$

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$$H_{03}^1 = \frac{1}{\beta \lambda} F_{04}^1 - \frac{16\beta^2 \sigma^{1/2} v^2}{3} \left[\lambda^3 \mu_1^2 + \lambda v^2 (2\sigma + 1) \right] \sqrt{\mu_1^2 - \beta^2 v^2}$$

As may be suspected from the complicated results generically represented by E_{mn} , F_{mn} , G_{mn} , and H_{mn} , the section lift and moment coefficients are not simplified to any large extent by algebraic combination. The lift and moment coefficients, as given in equation (B3), are in a form which is perhaps better for numerical evaluation than a combined form.

Some cautionary remarks are in order with regard to equations (B1) and (B3). In evaluating the coefficients L_i and M_i (where $i = 3, 4$), the terms involving k^2 should be ignored because the lift and moment coefficients associated with α have been treated completely only to the same order as those associated with h/U . In equation (B3) only real quantities are admitted. Therefore, the following restrictions must be noted (for $\frac{v}{\mu_1} \geq \frac{1}{\beta}$):

$$\begin{aligned} \sqrt{\mu_1^2 - \beta^2 v^2} &= 0 \\ \cos^{-1} \frac{\mu_1 + \beta^2 \lambda v}{\beta(\lambda \mu_1 + v)} &= 0 \\ \cos^{-1} \frac{\mu_1 - \beta^2 \lambda v}{\beta(\lambda \mu_1 - v)} &= \pi \\ \cosh^{-1} \frac{\mu_1}{\beta v} &= 0 \end{aligned}$$

Total Force and Moment Coefficients

As indicated in equations (20) and (21), the total lift and moment coefficients \bar{L}_1 and \bar{M}_1 for the delta wing may be obtained from the previously derived L_1 and M_1 by setting $\mu_1 = 1$ in L_1 and M_1 and integrating the resulting expressions over the span of the wing. The results of the spanwise integration combine readily to yield expressions which are much simpler than the corresponding ones in the section-coefficient case. Without further detail, the total force and moment coefficients are:

$$\left. \begin{aligned} \bar{L}_1 + i\bar{L}_2 &= \bar{L}_1' + i\bar{L}_2' \\ \bar{L}_3 + i\bar{L}_4 &= \bar{L}_3' + i\bar{L}_4' - \left(\frac{1}{k} + 2\mu_0\right)(\bar{L}_1' + i\bar{L}_2') \\ \bar{M}_1 + i\bar{M}_2 &= \bar{M}_1' + i\bar{M}_2' - 2\mu_0(\bar{L}_1' + i\bar{L}_2') \\ \bar{M}_3 + i\bar{M}_4 &= \bar{M}_3' + i\bar{M}_4' - 2\mu_0(\bar{L}_3' + i\bar{L}_4') - \left(\frac{1}{k} + 2\mu_0\right)(\bar{M}_1' + i\bar{M}_2') \end{aligned} \right\} \quad (B4)$$

where

$$\bar{L}_1' = \frac{2\lambda}{3\beta^3} - \frac{M^2\lambda}{15\beta^7}(4M^2 + 1)k^2$$

$$\bar{L}_2' = \frac{\lambda}{\beta k} - \frac{M^2\lambda}{2\beta^5}k$$

$$\bar{L}_3' = \frac{2\lambda}{3\beta^3}$$

$$\bar{L}_4' = \frac{4\lambda}{3\beta k} - \frac{2M^2\lambda}{5\beta^5}k$$

$$\bar{M}_1' = \frac{\lambda}{\beta^3} - \frac{M^2\lambda}{9\beta^7}(4M^2 + 1)k^2$$

$$\bar{M}_2' = \frac{4\lambda}{3\beta k} - \frac{4M^2\lambda}{5\beta^5} k$$

$$\bar{M}_3' = \frac{16\lambda}{15\beta^3}$$

$$\bar{M}_4' = \frac{2\lambda}{\beta k} - \frac{2M^2\lambda}{3\beta^5} k$$

If equations (22) and (23) had been put in the conventional form,

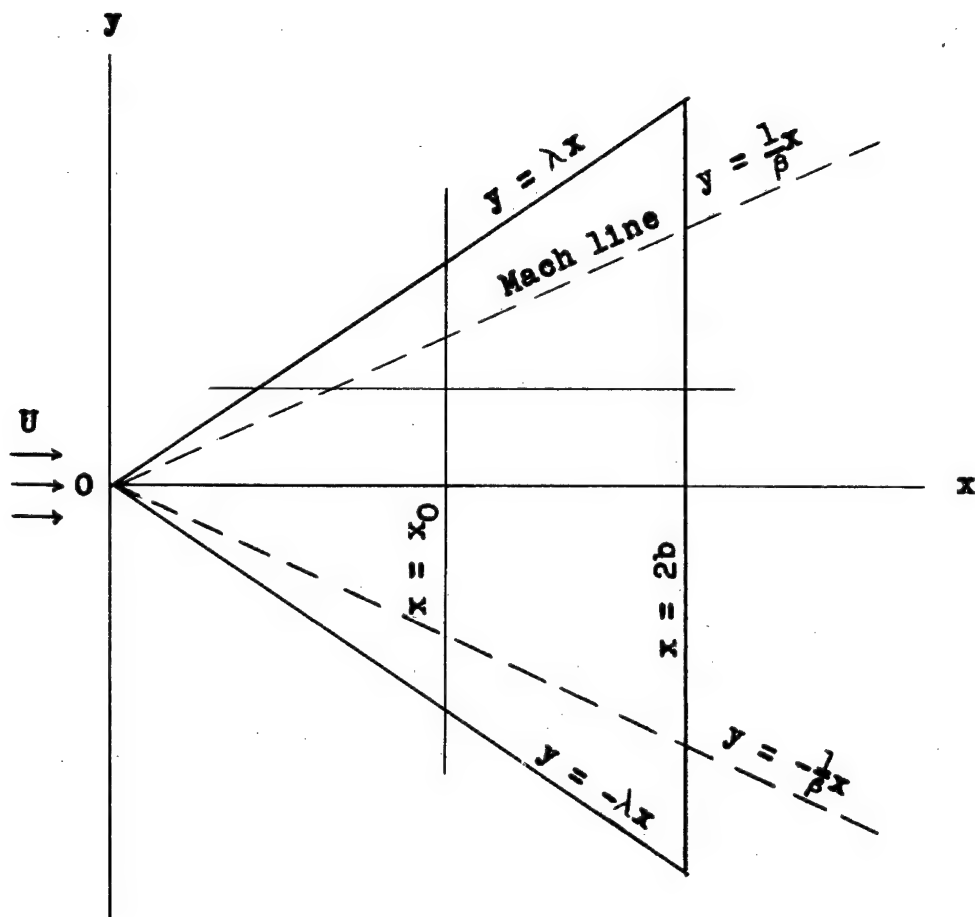
$$L = C_L q S$$

$$M = C_M q S c$$

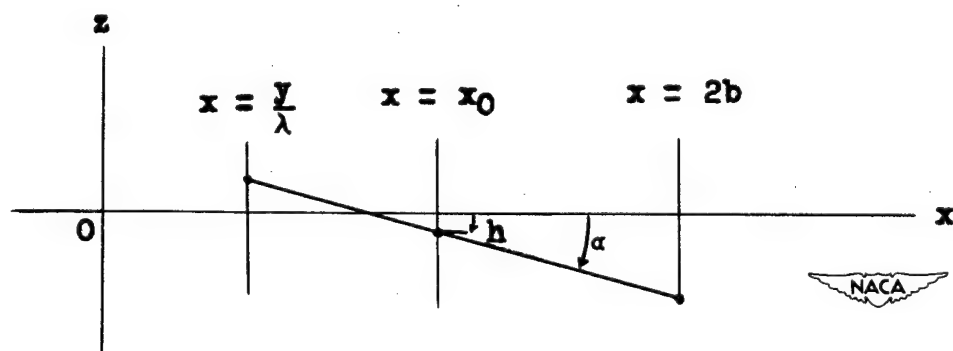
where S is the area of the wing (for the triangle $S = 4b^2\lambda$), q is the dynamic pressure, and c is the wing chord, then the lift and moment coefficients C_L and C_M , obtained from equation (B4), would be independent of λ . This result is rather remarkable since it means that the lift and moment coefficients for a triangular wing with supersonic edges are independent of the vertex angle of the triangle. The same observation was previously noted in reference 4.

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(a) Projection of plan form on xy-plane.



(b) Section y.

Figure 1.- Coordinate system and the two degrees of freedom α and h .

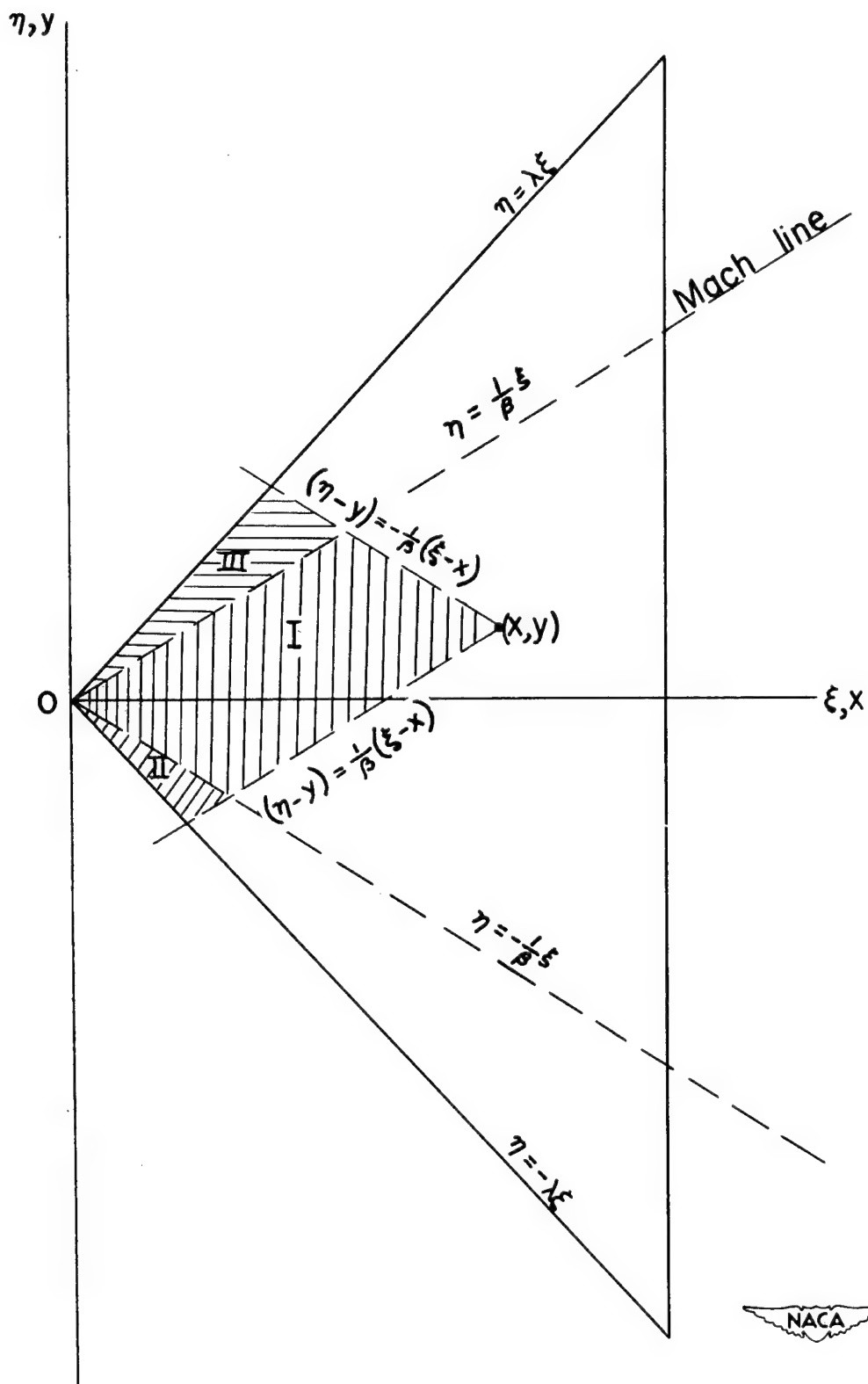


Figure 2.- Regions of integration for the velocity potential.

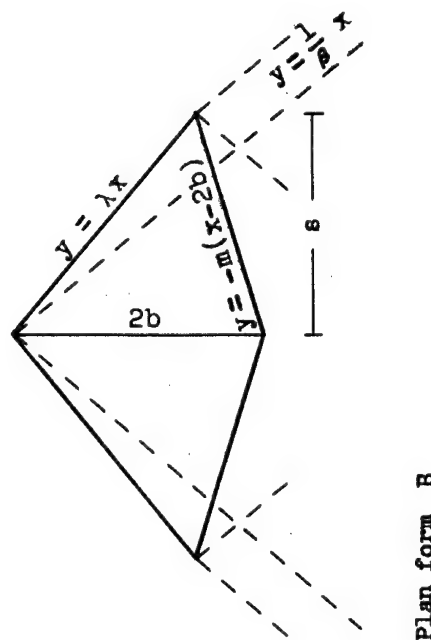
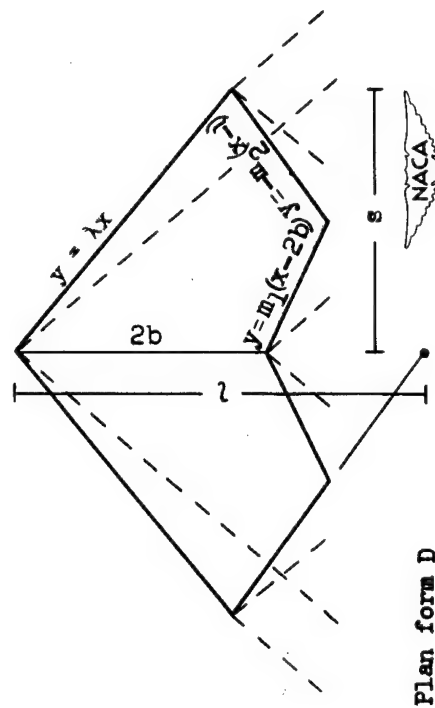
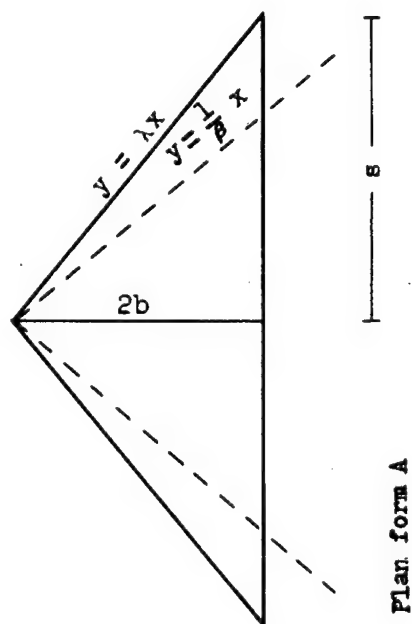
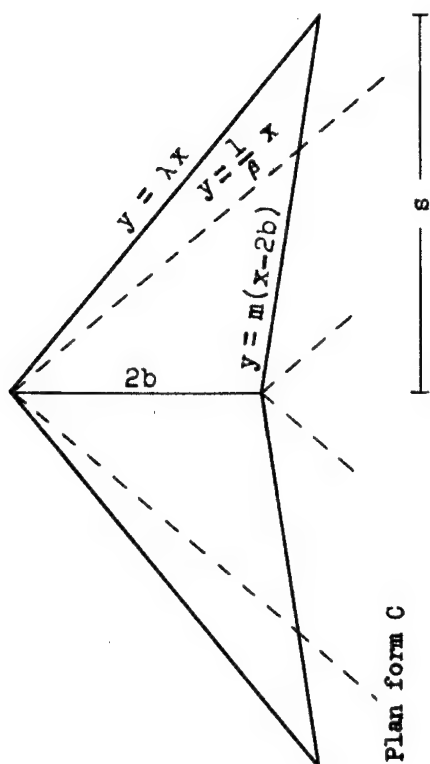
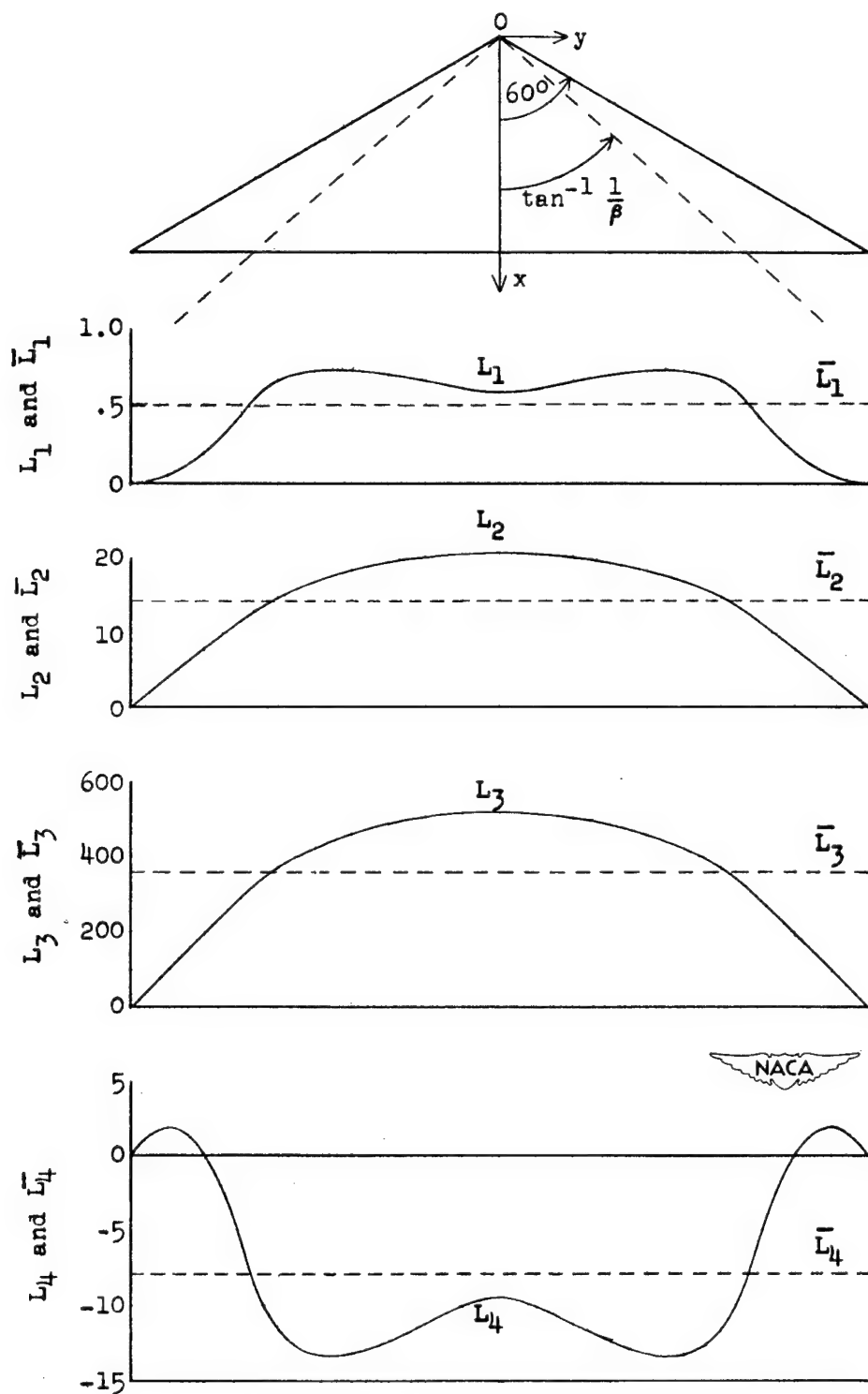
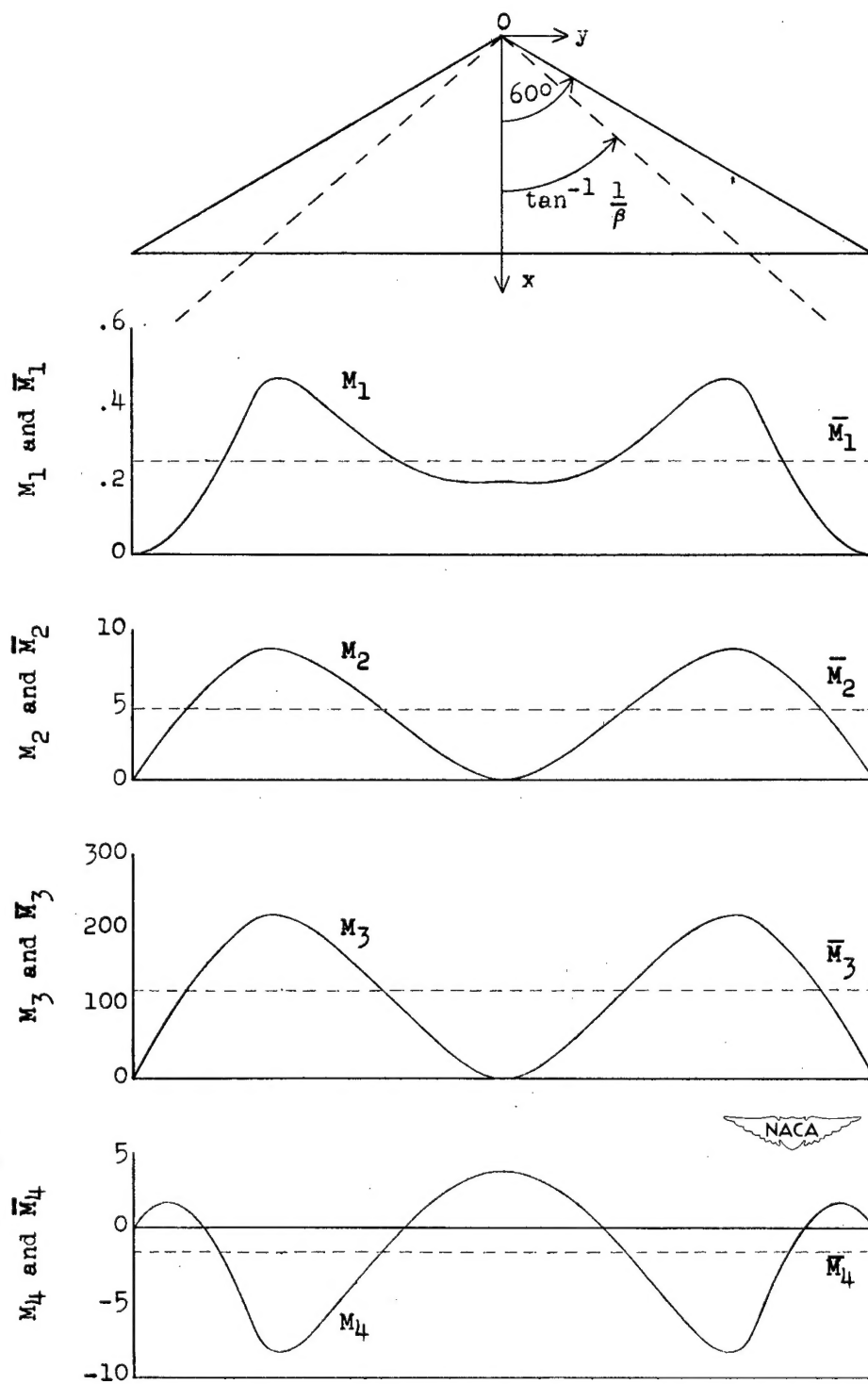


Figure 3.- Different plan forms for which the force equation (18) and the moment equation (19) apply.



(a) Lift force.

Figure 4.- Spanwise distribution of components of lift force and moment coefficients for $\mu_0 = 0.5$, $\lambda = \sqrt{3}$, $M^2 = 1.75$, and $k = 0.04$.



(b) Moment.

Figure 4.- Concluded.

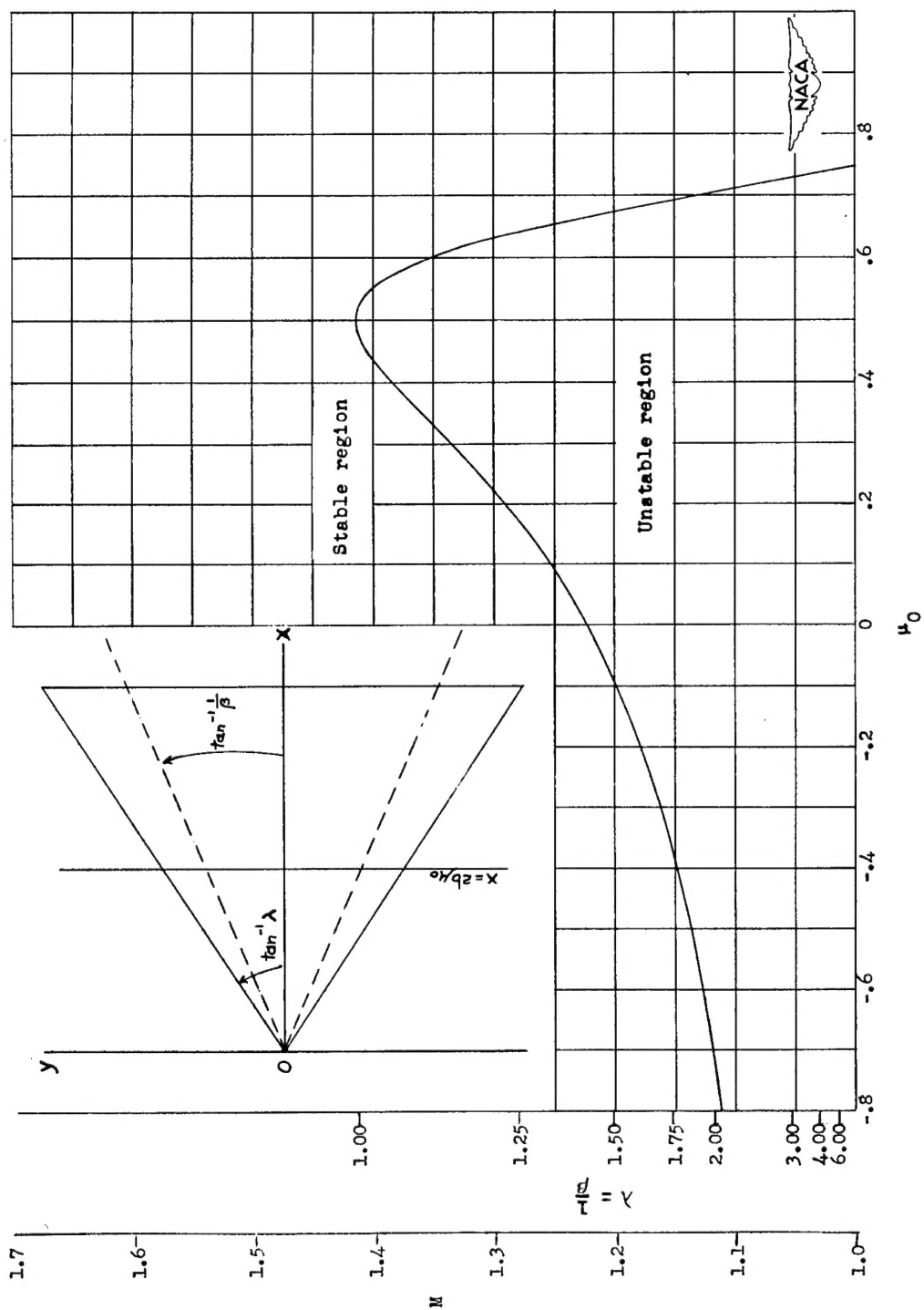






Figure 5.- Curve showing ranges of M and $\lambda = \frac{1}{\beta}$ against μ_0 for which the aerodynamic torsional damping moment vanishes for wide delta wings.

<p>NACA TN 2494</p> <p>National Advisory Committee for Aeronautics. LIFT AND MOMENT ON OSCILLATING TRIANGULAR AND RELATED WINGS WITH SUPERSONIC EDGES. Herbert C. Nelson. September 1951. 34p. diags. (NACA TN 2494)</p> <p>The linearized velocity potential for a thin, flat, triangular or related wing with supersonic edges, undergoing harmonic torsional oscillations and vertical translations in a constant supersonic stream, is derived in terms of a frequency parameter. Closed expressions for the velocity potential, section force and moment coefficients for the triangular wing are developed to the third power of the frequency of oscillation.</p> <p>Copies obtainable from NACA, Washington</p>	<p>1. Wings, Complete - Sweep (1.2.2.2.3)</p> <p>2. Mach Number Effects - Complete Wings (1.2.2.6)</p> <p>3. Vibration and Flutter - Wings and Alleronns (4.2.1)</p> <p>I. Nelson, Herbert C. II. NACA TN 2494</p> 	<p>NACA TN 2494</p> <p>National Advisory Committee for Aeronautics. LIFT AND MOMENT ON OSCILLATING TRIANGULAR AND RELATED WINGS WITH SUPERSONIC EDGES. Herbert C. Nelson. September 1951. 34p. diags. (NACA TN 2494)</p> <p>The linearized velocity potential for a thin, flat, triangular or related wing with supersonic edges, undergoing harmonic torsional oscillations and vertical translations in a constant supersonic stream, is derived in terms of a frequency parameter. Closed expressions for the velocity potential, section force and moment coefficients for the triangular wing are developed to the third power of the frequency of oscillation.</p> <p>Copies obtainable from NACA, Washington</p>	<p>1. Wings, Complete - Sweep (1.2.2.2.3)</p> <p>2. Mach Number Effects - Complete Wings (1.2.2.6)</p> <p>3. Vibration and Flutter - Wings and Alleronns (4.2.1)</p> <p>I. Nelson, Herbert C. II. NACA TN 2494</p> 
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